Journal of Mathematical Sciences: Advances and Applications
Volume 27, 2014, Pages 95-102

A CHARACTERIZATION OF INVERSE ALMOST HEMIRINGS

BERHANU ASSAYE and G. C. RAO

Department of Mathematics
Bahir Dar University
Bahir Dar
Ethiopia
e-mail: berhanu_assaye@yahoo.com

Department of Mathematics
Andhra University
Visakhapatnam - 530003
Andhra Pradesh
India
e-mail: gcraomaths@yahoo.co.in

Abstract

In this paper, we study the properties of inverse almost hemirings. We introduce the concept of inverse almost hemirings as an abstraction of inverse hemirings. We also introduce the concept of Clifford almost hemirings. Lastly, we characterize an almost hemiring as a Clifford almost hemiring.

1. Introduction

We introduced the concept of almost semiring in [2] and [3] as an abstraction of semiring. An algebra $(S, +, \cdot)$ of type (2, 2) is called an almost semiring, if it satisfies the following axioms:

2010 Mathematics Subject Classification: 06D99, 06D15.

Keywords and phrases: almost hemiring, K-ideal, inverse almost hemiring, Clifford almost hemiring.

Received April 7, 2014; Revised April 11, 2014

(i) $(ab)c = a(bc)$,

(ii) $(a + b)c = ac + bc$,

(iii) $[(a + b) + c]d = [a + (b + c)]d$,

for each $a, b, c, d \in S$.

If, in addition to this, + is commutative, we say that $(S, +, \cdot)$ is additively commutative almost semiring. If + is associative, we say $(S, +, \cdot)$ is an associative almost semiring. It can be verified by means of examples that the axioms in the definition of almost semiring are independent of one another. Every semiring is an almost semiring but the converse is not true.

An additively commutative almost semiring S with 0 satisfying $x \cdot 0 = 0 \cdot x = 0$ for every x in S is said to be an almost hemiring. Let $(S, +)$ be an additive group of characterstic two. Suppose S contains at least four elements. Let us define multiplication $\cdot$ on S by

$$a \cdot b = \begin{cases} a, & \text{if } b \neq 0, \\ 0, & \text{if } b = 0. \end{cases}$$

Then $(S, +, \cdot)$ is an almost hemiring [1].

Semiring is an algebraic structure $(S, +, \cdot)$ consisting of a non empty set S together with two binary operations + and $\cdot$ on S such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by ring like distributivity as defined by Ghosh in [6]. Kervellas in [5] defined a semiring S as an inverse semiring (sometimes used the term additively inverse semiring) if and only if $(S, +)$ is an inverse semigroup, i.e., if and only if for each $a \in S$, there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. Also, Kervellas in [5] defined an (additively) inverse semiring. Clifford semirings and K-ideals are defined by Sen et al. in [4].

In this paper, we introduce the concept of inverse almost hemiring as a generalization of inverse hemiring and Clifford almost hemiring as an abstraction of Clifford hemiring. We also characterize an inverse almost hemiring as a Clifford almost hemiring.

2. Inverse Almost Hemirings

A semiring S is called an inverse semiring ([5, p. 278] used the term additively inverse semiring) if and only if $(S, +)$ is an inverse semigroup, i.e., if and only if for each $a \in S$, there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. In view of this, we define an inverse almost semiring.

Definition 2.1. Let S be an associative almost hemiring. Then S is called an *inverse almost hemiring*, if

(1) $(S, +)$ is an inverse semigroup,

(2) $(ab)' = ab'$ for all $a, b \in S$,

(3) $a(b + b') = ab + ab'$ for all $a, b \in S$.

There are almost hemirings, which are not hemirings that fail to satisfy $(ab)' = ab'$ and $a(b + b') = ab + ab'$ for all $a, b \in S$. Let us consider the following example:

Example 2.2. Let us take the additive group of integers modulo six $(Z_6, \oplus_6)$. Define multiplication $\cdot$ on Z_6 by

$$a \cdot b = \begin{cases} a, & \text{if } b \neq 0, \\ 0, & \text{if } b = 0. \end{cases}$$

Then $(Z_6, \oplus_6, \cdot)$ is an almost semiring with an inverse semigroup $(Z_6, \oplus_6)$ but it fails to satisfy $(ab)' = ab'$ and $a(b + b') = ab + ab'$ for all $a, b \in Z_6$.

Also there are almost hemirings which are not semirings, which meet the requirements to be the inverse almost hemirings.

Example 2.3. Let $(G, +)$ be an additive group of characteristic two. Assume G has at least four elements. Let us define multiplication $\cdot$ on G by

$$a \cdot b = \begin{cases} a, & \text{if } \; b \neq 0, \\ 0, & \text{if } \; b = 0. \end{cases}$$

Then $(G, +, \cdot)$ is an inverse almost hemiring with $a' = a$.

Definition 2.4. Let S be an almost hemiring an ideal I of S is called a *K-ideal* if, for $a \in S$, $b \in I$, either $a + b \in I$ or $b + a \in I$, then $a \in I$.

Let us take S as inverse almost hemiring from now on in this section unless justified.

Lemma 2.5. $E^{+}(S) = \{a + a' : a \in S\}$.

Proof. Let $D = \{a + a' : a \in S\}$. We prove that $D = E^{+}(S)$. Let $a + a' \in D$. Then

$$(a + a') + (a + a') = (a + a' + a) + a' = a + a'.$$

This implies that $a + a' \in E^{+}(S)$. Hence $D \subseteq E^{+}(S)$.

Conversely, assume that $a \in E^{+}(S)$, then $a + a = a$.

$$\Rightarrow a + a + a = a, \; a + a' + a = a, \text{ and } a' + a + a' = a'$$

$$\Rightarrow a' + a = a = a' \in E^{+}(S)$$

$$\Rightarrow a + a' = a = a' \in D.$$

Thus $E^{+}(S) \subseteq D$ and hence $D = E^{+}(S)$.

Lemma 2.6. *S satisfies the following conditions*:

(1) $(a + a')a = a + a'$,

(2) $(a + a')(b + b') = (b + b')(a + a')$,

(3) $a + (a + a')b = a$ for all $a, b \in S$,

if and only if $E^+(S)$ is a distributive lattice.

Proof. Let us assume the conditions. Let $e \in E^+(S)$. Then $e = e'$ by Lemma 2.5.

Idempotency holds on $E^+(S)$ since $e^2 = (e + e)e = (e + e')e = e[\because (1)]$.

Commutativity also holds on $E^+(S)$ since for $e, f \in E^+(S)$. We have

$$ef = (e + e')(f + f') = (f + f')(e + e')[\because (2)].$$

This implies that $ef = fe$.

Also one can easily show multiplication is associative on $E^+(S)$.

Now, to prove $(E^+(S), +, \cdot)$ is a distributive lattice, it is enough to show multiplication is distributive over addition on $E^+(S)$.

Let $e, f, g \in E^+(S)$. Then $(e + f)g = g(e + f)$ and $eg + fg = ge + gf$ $[\because (2)]$. And also the absorption property $(e + f)e = (f + e)e = e + ef = e$ holds true since $e + (e + e')f = e + ef = e[\because (3)]$. Therefore $(E^+(S), +, \cdot)$ is a distributive lattice.

Conversely, assume that $(E^+(S), +, \cdot)$ is a distributive lattice. Let $a, b \in S$. Then $(a + a'), (b + b') \in (E^+(S))$.

Hence

$$\begin{aligned}(a + a')(b + b') &= a(b + b') + a'(b + b')\\ &= ab + ab' + a'b + a'b'\\ &= ab + (ab)' + (ab)' + ab\\ &= ab + (ab)'\\ &= ab + a'b\\ &= (a + a')b.\end{aligned}$$

Now

(1) $(a + a')(a + a') = (a + a')a = (a + a')$.

(2) $(a + a')(b + b') = (b + b')(a + a')$.

(3) $(a + a') + (a + a')(b + b') = (a + a')$.

This implies $a + a' + a + (a + a')b = a + a' + a$.

Thus $a + (a + a')b = a$.

Lemma 2.7. *$E^+(S)$ is a K-ideal of S if and only if $a \in S$, $a + b = b$ for some $b \in S$ implies $a + a = a$.*

Proof. Let $E^+(S)$ be a K-ideal of S. Assume $a \in S$, $a + b = b$ for some $b \in S$, then $a + b + b' = b + b'$.

$\Rightarrow a + (b + b') \in E^+(S)$

$\Rightarrow a \in E^+(S)$ since $E^+(S)$ is a K-ideal of S. Thus $a + a = a$.

Conversely, assume the condition holds. Let $e, f \in E^+(S)$ and $a \in S$ such that $a + e = f$, then

$$a + e + e + f = f + e + f$$
$$= e + f.$$

This implies $a + (e + f) = e + f$ and hence $a + a = a$ follows by hypothesis.

Definition 2.8. S is called *Clifford almost hemiring*, if $E^{+}(S)$ is a distributive lattice as well as a K-ideal of S.

There are Clifford almost hemirings which are not hemirings. For this, consider the following example:

Example 2.9. Let D be a distributive lattice. Assume that $(G, +, \cdot)$ is an almost hemiring defined in Example 2.3, then $S = D \times G$ with pointwise operations is a Clifford almost hemiring since $E^{+}(S) = D \times \{0_g\} \cong D$, where 0_g is the zero G.

We give only the statement of the following theorem since its proof follows directly from Lemmas 2.6, 2.7 and Definition 2.8.

Theorem 2.10. *S is Clifford almost hemiring if and only if it satisfies the following conditions: For all $a, b, c \in S$,*

(1) $(a + a')a = a + a'$,

(2) $(a + a')(b + b') = (b + b')(a + a')$,

(3) $a + (a + a')b = a$,

(4) $a \in S$, $a + c = c$ *for some* $c \in S$ *implies* $a + a = a$.

References

[1] G. C. Rao and Berhanu Assaye, A fundamental theorem of homomorphisms on almost hemirings, Communicated to Chamchuri Journal of Mathematics.

[2] G. C. Rao and Berhanu Assaye, Almost semirings-I, South East Asian Bulletin of Mathematics 36 (2012), 135-141.

[3] G. C. Rao and Berhanu Assaye, Almost semirings-II, South East Asian Bulletin of Mathematics 37 (2013), 229-244.

[4] M. K. Sen, S. K. Maity and K. P. Shaum, Clifford semirings and generalized Clifford semirings, Taiwanese Journal of Mathematics 9(3) (2005), 433-444.

[5] P. H. Karvellas, Inverse semirings, J. Austral. Math. Soc. 18 (1974), 277-288.

[6] Shamik Ghosh, A characterization of semirings which are subdirect products of a distributive lattice and a ring, Semigroup Forum 59 (1999), 106-120.

■